

# Tensor product of left polaroid operators

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## Abstract

A Banach space operator  $T \in B(\mathcal{X})$  is left polaroid if for each  $\lambda \in \text{iso } \sigma_a(T)$  there is an integer  $d(\lambda)$  such that  $\text{asc}(T - \lambda) = d(\lambda) < \infty$  and  $(T - \lambda)^{d(\lambda)+1}\mathcal{X}$  is closed;  $T$  is finitely left polaroid if  $\text{asc}(T - \lambda) < \infty$ ,  $(T - \lambda)\mathcal{X}$  is closed and  $\dim(T - \lambda)^{-1}(0) < \infty$  at each  $\lambda \in \text{iso } \sigma_a(T)$ . The left polaroid property transfers from  $A$  and  $B$  to their tensor product  $A \otimes B$ , hence also from  $A$  and  $B^*$  to the left-right multiplication operator  $\tau_{AB}$ , for Hilbert space operators; an additional condition is required for Banach space operators. The finitely left polaroid property transfers from  $A$  and  $B$  to their tensor product  $A \otimes B$  if and only if  $0 \notin \text{iso } \sigma_a(A \otimes B)$ ; a similar result holds for  $\tau_{AB}$  for finitely left polaroid  $A$  and  $B^*$ .

## 1. Introduction

A Banach space operator  $T \in B(\mathcal{X})$  is *polar* at a point  $\lambda$  in its spectrum  $\sigma(T)$  if  $T - \lambda I$  has both finite ascent  $\text{asc}(T - \lambda I)$  and descent  $\text{dsc}(T - \lambda I)$ . Apparently, if  $T$  is polar at  $\lambda \in \sigma(T)$ , then  $\lambda \in \text{iso } \sigma(T)$ , the set of isolated points of  $\sigma(T)$ . We say that  $T$  is *polaroid* if  $T$  is polar at every  $\lambda \in \text{iso } \sigma(T)$ . Given Banach spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , let  $\mathcal{X} \overline{\otimes} \mathcal{Y}$  denote the algebraic completion, endowed with a reasonable uniform cross-norm, of the tensor product of  $\mathcal{X}$  and  $\mathcal{Y}$ . It is known, [8, Theorem 3], that the polaroid property transfers from  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  to their tensor product  $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ .

$T \in B(\mathcal{X})$  is *left polar* (respectively, *right polar*) of order  $d$  at a point  $\lambda$  in its approximate point spectrum  $\sigma_a(T)$  (respectively, surjectivity spectrum  $\sigma_s(T)$ ) if  $\text{asc}(T - \lambda I) = d < \infty$  and  $(T - \lambda I)^{d+1}(\mathcal{X})$  is closed (respectively,  $\text{dsc}(T - \lambda I) = d < \infty$  and  $(T - \lambda I)^d \mathcal{X}$  is closed). It is known that if  $T$  is left polar (respectively, right polar) at  $\lambda$ , then  $\lambda \in \text{iso } \sigma_a(T)$  (respectively,  $\lambda \in \text{iso } \sigma_s(T)$ ). We say that  $T$  is *left polaroid* (respectively, *right polaroid*) if  $T$  is left polar (respectively, right polar) at every  $\lambda \in \text{iso } \sigma_a(T)$  (respectively,  $\lambda \in \text{iso } \sigma_s(T)$ ). Apparently,  $T$  is right polaroid if and only if  $T^*$  is left polaroid,  $T$  is polaroid if it is both left and right polaroid and a polaroid operator  $T$  is both left and right polaroid whenever  $\text{iso } \sigma(T) = \text{iso } \sigma_a(T) \cup \text{iso } \sigma_s(T)$ . The question that we consider here is the following: Does the left polaroid property transfer from  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  to  $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$ ? The answer to this question is a yes in the case in which  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces. In the general case, if  $A$  and  $B$  are left polar (of order  $d(\lambda)$  and  $d(\mu)$  at points  $\lambda \in \text{iso } \sigma_a(A)$  and  $\mu \in \text{iso } \sigma_a(B)$ ), and if the closed subspaces  $(A - \lambda I)\mathcal{X} + (A - \lambda I)^{-d(\lambda)}(0)$  and  $(B - \mu I)\mathcal{Y} + (B - \mu I)^{-d(\mu)}(0)$  are complemented in  $\mathcal{X}$  and  $\mathcal{Y}$  respectively for every  $\lambda \in \text{iso } \sigma_a(A)$  and  $\mu \in \text{iso } \sigma_a(B)$ , then  $A \otimes B$  is left polaroid.

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A stronger version of the left polaroid property says that  $T \in B(\mathcal{X})$  is *finitely left polaroid* if  $T$  is left polar and  $\alpha(T - \lambda I) = \dim(T - \lambda I)^{-1}(0) < \infty$  at every  $\lambda \in \text{iso } \sigma_a(T)$ . The finitely left polaroid property transfers from  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  to  $A \otimes B$  if and only if  $0 \notin \text{iso } \sigma_a(A \otimes B)$ . We characterize  $\sigma_a(A \otimes B)$  in terms of the set of finite left poles and of the Browder essential approximate point spectrum of  $A$  and of  $B$ , see section 4.

Similar results will be proved for the elementary operator  $\tau_{AB} = L_A R_B$  both in the case of left polaroid operators and of finitely left polaroid operators  $A$  and  $B^*$ .

## 2. Preliminaries

Unless otherwise stated, from now on  $\mathcal{X}$  (similarly,  $\mathcal{Y}$ ) shall denote a complex Banach space and  $B(\mathcal{X})$  (similarly,  $B(\mathcal{Y})$ ) the algebra of all bounded linear maps defined on and with values in  $\mathcal{X}$  (respectively,  $\mathcal{Y}$ ). Henceforth, *we shall reserve the symbols  $T$  and  $S$  for general Banach space operators, and the symbols  $A$  and  $B$  for operators  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$* . Given  $T \in B(\mathcal{X})$ ,  $T^* \in B(\mathcal{X}^*)$  shall denote the adjoint of  $T$ , where  $\mathcal{X}^*$  is the dual space of  $\mathcal{X}$ . Recall that  $T \in B(\mathcal{X})$  is said to be *bounded below*, if  $T^{-1}(0) = \{0\}$  and the range  $T\mathcal{X}$  of  $T$  is closed. Denote the *approximate point spectrum* of  $T$  by  $\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not bounded below}\}$ , where  $T - \lambda$  stands for  $T - \lambda I$ ,  $I$  the identity map of  $B(\mathcal{X})$ . Let  $\sigma_s(T) = \{\lambda \in \mathbb{C} : (T - \lambda)\mathcal{X} \neq \mathcal{X}\}$  denote the *surjectivity spectrum* of  $T$ . Clearly,  $\sigma_a(T) \cup \sigma_s(T) = \sigma(T)$ , the spectrum of  $T$ .

Given  $T \in B(\mathcal{X})$ , if  $T\mathcal{X}$  is closed and  $\alpha(T) = \dim T^{-1}(0)$  (resp.,  $\beta(T) = \dim \mathcal{X}/T\mathcal{X}$ ) is finite, then  $T$  is said to be *upper semi-Fredholm* (resp., *lower semi-Fredholm*). Moreover, such an operator has a well defined *index*, i.e.,  $\text{ind}(T) = \alpha(T) - \beta(T)$ . Naturally, from this class of operators the *upper semi-Fredholm spectrum* can be derived, i. e., the set

$$\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\}.$$

The lower semi-Fredholm spectrum can be defined in a similar way and it will be denoted by  $\sigma_{SF_-}(T)$ .

Let  $T \in B(\mathcal{X})$ . Recall that  $\text{asc}(T)$  (respectively,  $\text{dsc}(T)$ ) is the least non-negative integer  $n$  such that  $T^{-n}(0) = T^{-(n+1)}(0)$  (respectively,  $T^n\mathcal{X} = T^{n+1}\mathcal{X}$ ); if no such integer exists, then  $\text{asc}(T)$  (respectively,  $\text{dsc}(T)$ ) is infinite. Recall also that  $\text{asc}(T - \lambda) < \infty \implies T$  has the single-valued extension property at  $\lambda$ , and that if  $T - \lambda$  is upper semi-Fredholm and has the single-valued extension property (at 0) then  $\text{asc}(T - \lambda) < \infty$ . Here,  $T$  has the single-valued extension property at  $\lambda$ , shortened henceforth to SVEP at  $\lambda$ , if, for every open neighbourhood  $\mathcal{U}$  of  $\lambda$ , the only analytic function  $f : \mathcal{U} \rightarrow \mathcal{X}$  satisfying  $(T - \lambda)f(\lambda) = 0$  is the function  $f \equiv 0$ . We say that  $T$  has SVEP on a subset of the complex plane  $\mathbb{C}$  if it has SVEP at every point of the subset.

The *Weyl essential approximate point spectrum* and the *Browder essential approximate point spectrum* of  $T \in B(\mathcal{X})$  are the sets

$$\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \text{ is not upper semi-Fredholm or } 0 < \text{ind}(T - \lambda)\}$$

and

$$\sigma_{ab}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{aw}(T) \text{ or } \text{asc}(T - \lambda) = \infty\},$$

respectively. It is clear that

$$\sigma_{SF_+}(T) \subseteq \sigma_{aw}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_a(T).$$

Concerning the main properties of the aforementioned spectra, see [1, 14, 15, 16].

We say that  $T \in B(\mathcal{X})$  is *semi B-Fredholm*,  $T \in \Phi_{SBF}(\mathcal{X})$ , if there exists a non-negative integer  $n$  such that  $T^n \mathcal{X}$  is closed and the induced operator  $T_{[n]} = T|_{T^n \mathcal{X}}$  ( $T_{[0]} = T$ ) is semi-Fredholm, upper or lower, in the usual sense. Observe that  $T_{[m]}$  is then semi-Fredholm for all  $m \geq n$ : we define the *index* of  $T$  by  $\text{ind}(T) = \text{ind}(T_{[n]})$ . Let

$$\Phi_{SBF_+^-}(\mathcal{X}) = \{T \in \Phi_{SBF}(\mathcal{X}) : T \text{ is upper semi B-Fredholm with } \text{ind}(T) \leq 0\};$$

then the upper semi B-Weyl spectrum of  $T$  is the set

$$\sigma_{UBW}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \notin \Phi_{SBF_+^-}(\mathcal{X})\}.$$

The lower semi B-Weyl spectrum can be defined in a similar way and it will be denoted by  $\sigma_{LBW}(T)$ . In addition,  $T$  will be said to be *B-Weyl*, if  $T$  is both upper and lower semi B-Fredholm (equivalently  $T$  is *B-Fredholm*) and  $\text{ind}(T) = 0$ . The *B-Weyl* spectrum of  $T$  is the set

$$\sigma_{BW}(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not B-Fredholm or } \text{ind}(T - \lambda) \neq 0\}.$$

Note that  $\sigma_{LBW}(T) = \sigma_{UBW}(T^*)$  and  $\sigma_{BW}(T) = \sigma_{UBW}(T) \cup \sigma_{LBW}(T)$ .

We say that  $T$  is *quasi-Fredholm of degree  $d$*  ( $\geq 0$ ), if  $\dim(T^n \mathcal{X} \cap T^{-1}(0)) \setminus (T^{n+1} \mathcal{X} \cap T^{-1}(0)) = 0$  for all  $n \geq d$ , and the subspaces  $T^{-d}(0) + T\mathcal{X}$  and  $T^{-1}(0) \cap T^d \mathcal{X}$  are closed. Every semi B-Fredholm operator is quasi-Fredholm [5].

Let  $\Pi^\ell(T)$  denote the set of left poles of  $T \in B(\mathcal{X})$ , i.e.,  $\Pi^\ell(T) = \{\lambda \in \sigma_a(T) : \text{asc}(T - \lambda) = d < \infty \text{ and } (T - \lambda)^{d+1} \mathcal{X} \text{ is closed}\}$ . If  $\lambda \in \Pi^\ell(T)$  is a left pole of order  $d$ , then  $\lambda \in \text{iso } \sigma_a(T)$ ,  $\lambda \notin \sigma_{UBW}(T)$  ([9, Lemma 3.1]), and  $(T - \lambda)_{[d]} = (T - \lambda)|_{(T - \lambda)^d \mathcal{X}}$  is bounded below ([3, Theorem 2.5]), where if  $M \subseteq \mathbb{C}$ , then  $\text{acc } M$  stands for the set of limit points of  $M$  and  $\text{iso } M = M \setminus \text{acc } M$ . (Indeed,  $\lambda \in \Pi^\ell(T)$  if and only if  $\lambda \notin \sigma_{UBW}(T)$  and  $T$  has SVEP at  $\lambda$ .) Furthermore, if we let

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}$$

denote the quasi-nilpotent part of  $T$ , then  $H_0(T - \lambda) = (T - \lambda)^{-d}(0)$  ([2, Theorem 2.3]).

It is known, [9, Lemma 3.5], that if  $T^*$  has SVEP at points  $\lambda \notin \sigma_{UBW}(T)$ , then  $\sigma_{UBW}(T) = \sigma_{BW}(T)$ . This implies that if  $\lambda \in \Pi^\ell(T)$  and  $T^*$  has SVEP at points  $\lambda \notin \sigma_{UBW}(T)$ , then  $\lambda \in \text{iso } \sigma(T)$  and  $T$  is polar at  $\lambda$  ([9, Corollary 3.13]). Consequently, if  $T^*$  has SVEP at points  $\lambda \in \Pi^\ell(T)$ , then  $\Pi^\ell(T) = \Pi(T) = \{\lambda \in \text{iso } \sigma(T) : T \text{ is polar at } \lambda\}$ .

Concerning finitely left polaroid operators, recall from [10, Theorem 3.8] that if  $T \in B(\mathcal{X})$ ,  $\alpha(T) < \infty$  and  $\text{asc}(T) < \infty$ , then  $T^n \mathcal{X}$  is closed for some integer  $n > 1$  if and only if  $T\mathcal{X}$  is closed. Hence  $T$  is finitely left polaroid if and only if  $\alpha(T - \lambda I) < \infty$ ,  $\text{asc}(T - \lambda I) < \infty$  and  $(T - \lambda I)\mathcal{X}$  is closed for every  $\lambda \in \text{iso } \sigma_a(T)$ . Let  $\Pi_0^\ell(T)$  denote the set of finite left poles of  $T$ , i.e.,

$$\Pi_0^\ell(T) = \{\lambda \in \text{iso } \sigma_a(T) : T - \lambda \text{ is upper semi-Fredholm and } \text{asc}(T - \lambda) < \infty\}.$$

Then  $T \in B(\mathcal{X})$  is finitely left polaroid if and only if  $\text{iso } \sigma_a(T) = \Pi_0^\ell(T)$ .

In the following remark, several properties of finite left poles will be recalled.

**Remark 2.1.** Let  $T \in B(\mathcal{X})$ . Then  $\sigma_a(T) \setminus \sigma_{ab}(T) = \Pi_0^\ell(T)$  ([16, Corollary 2.2]). Additionally, if  $\lambda \in \text{iso } \sigma_a(T)$ , then the following statements are equivalent:

$$(i) \lambda \in \sigma_a(T) \setminus \sigma_{ab}(T), \quad (ii) \lambda \in \sigma_a(T) \setminus \sigma_{aw}(T).$$

As a result, if we let  $I_0^a(T) = \text{iso } \sigma_a(T) \setminus \Pi_0^\ell(T)$ , then (since  $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$ ),

$$I_0^a(T) \subseteq \sigma_{aw}(T) \subseteq I_0^a(T) \cup \text{acc } \sigma_a(T) = \sigma_{ab}(T)$$

([16, Theorem 2.1 and Corollary 2.3]). Therefore, necessary and sufficient for  $T$  to be finitely left polaroid is that one of the following statements holds:

- (iii)  $\sigma_{ab}(T) = \text{acc } \sigma_a(T)$ , (iv)  $\sigma_{aw}(T) = \text{acc } \sigma_a(T) \cap \sigma_{aw}(T)$ ,
- (v)  $\sigma_a(T) = \sigma_{ab}(T) \cup \text{iso } \sigma_a(T)$ ,  $\sigma_{ab}(A) \cap \text{iso } \sigma_a(A) = \emptyset$ .

Note that if  $\sigma_a(T) = \{0\}$ , in particular if  $T$  is a quasi-nilpotent operator, then  $T$  is not finitely left polaroid. In fact, if  $\sigma_a(T) = \{0\}$  and  $T$  is finitely left polaroid, then

$$\emptyset = \text{acc } \sigma_a(T) = \sigma_{ab}(T).$$

Since  $\sigma_{ab}(T) \neq \emptyset$  ([16, Corollary 2.4] and [14, Theorem 1]), this is a contradiction.

Let  $\mathcal{X} \overline{\otimes} \mathcal{Y}$  denote the completion of the algebraic tensor product of  $\mathcal{X}$  and  $\mathcal{Y}$ ,  $\mathcal{X} \otimes \mathcal{Y}$ , relative to some reasonable cross norm; let  $A \otimes B \in B(\mathcal{X} \overline{\otimes} \mathcal{Y})$  denote the tensor product of  $A$  and  $B$ . Then, [7, Lemma 5],

$$\sigma_{ab}(A \otimes B) = \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B).$$

Again, if  $\tau_{AB} = L_A R_B \in B(B(\mathcal{Y}, \mathcal{X}))$  denotes the elementary operator

$$\tau_{AB}(X) = L_A R_B(X) = A X B,$$

then

$$\sigma_{ab}(\tau_{AB}) = \sigma_a(A)\sigma_{ab}(B^*) \cup \sigma_{ab}(A)\sigma_a(B^*),$$

[6, Proposition 4.3 (iv)].

The following lemma studies the sets of the limit and the isolated points of the operators considered in this article.

**Lemma 2.2.** *If  $A$  and  $B$  are finitely left polaroid, then the following statements hold.*

- (i)  $\text{acc } \sigma_a(A \otimes B) \subseteq \sigma_{ab}(A \otimes B) \subseteq \text{acc } \sigma_a(A \otimes B) \cup \{0\}$ ;
- (ii)  $\text{iso } \sigma_a(A \otimes B) \setminus \{0\} \subseteq \Pi_0^l(A) \cdot \Pi_0^l(B) \subseteq \text{iso } \sigma_a(A \otimes B) \cup \{0\}$ .

*If, instead,  $A$  and  $B^*$  are finitely left polaroid, then the following statements hold.*

- (iii)  $\text{acc } \sigma_a(\tau_{AB}) \subseteq \sigma_{ab}(\tau_{AB}) \subseteq \text{acc } \sigma_a(\tau_{AB}) \cup \{0\}$ ;
- (iv)  $\text{iso } \sigma_a(\tau_{AB}) \setminus \{0\} \subseteq \Pi_0^l(A) \cdot \Pi_0^a(B^*) \subseteq \text{iso } \sigma_a(\tau_{AB}) \cup \{0\}$ .

*Proof.* Since  $\text{iso } \sigma_a(A) = \Pi_0^l(A)$  and  $\text{iso } \sigma_a(B) = \Pi_0^l(B)$  (Remark 2.1), the proof of (i) and (ii) is immediate from [11, Theorem 6] once one observes that  $\sigma_a(A \otimes B) = \sigma_a(A) \cdot \sigma_a(B)$  ([12, Theorem 4.4]),  $\sigma_{ab}(A) = \text{acc } \sigma_a(A)$ ,  $\sigma_{ab}(B) = \text{acc } \sigma_a(B)$  (Remark 2.1), and  $\sigma_{ab}(A \otimes B) = \sigma_a(A)\sigma_{ab}(B) \cup \sigma_{ab}(A)\sigma_a(B)$  ([7, Lemma 5]).

One argues similarly to prove (iii) and (iv): observe that  $\text{iso } \sigma_a(A) = \Pi_0^l(A)$  and  $\text{iso } \sigma_a(B^*) = \Pi_0^l(B^*)$ ,  $\sigma_a(\tau_{AB}) = \sigma_a(A) \cdot \sigma_a(B^*)$  ([6, Proposition 4.3 (i)]) and  $\sigma_{ab}(\tau_{AB}) = \sigma_a(A)\sigma_{ab}(B^*) \cup \sigma_{ab}(A)\sigma_a(B^*)$  ([6, Proposition 4.3 (iv)]).  $\square$

**Remark 2.3.** Note that under the conditions of Lemma 2.2

$$\sigma_{ab}(A \otimes B) = \sigma_{ab}(A) \cdot \sigma_{ab}(B) \cup \sigma_{ab}(A) \cdot \Pi_0^l(B) \cup \Pi_0^l(A) \cdot \sigma_{ab}(B)$$

([7, Lemma 5]). Similarly

$$\sigma_{ab}(\tau_{AB}) = \sigma_{ab}(A) \cdot \sigma_{ab}(B^*) \cup \sigma_{ab}(A) \cdot \Pi_0^l(B^*) \cup \Pi_0^l(A) \cdot \sigma_{ab}(B^*)$$

([6, Proposition 4.3 (iv)]).

### 3. Left polaroid operators

We say that a left polar operator  $T \in B(\mathcal{X})$ , of order  $d(\lambda)$  at  $\lambda \in \text{iso } \sigma_a(T)$ , satisfies property  $(\mathcal{P})$  if the closed subspace  $(T - \lambda)^{-d(\lambda)}(0) + (T - \lambda)\mathcal{X}$  is complemented in  $\mathcal{X}$  for every  $\lambda \in \text{iso } \sigma_a(T)$ . The following lemma proves that left polaroid operators satisfying property  $\mathcal{P}$  have a Kato type decomposition.

**Lemma 3.1.** *If  $T \in B(\mathcal{X})$  is left polaroid and satisfies property  $(\mathcal{P})$ , then for every  $\lambda \in \text{iso } \sigma_a(T)$  there exist  $T$ -invariant closed subspaces  $E_1$  and  $E_2$  such that  $\mathcal{X} = E_1 \oplus E_2$ ,  $H_0(T - \lambda) = (T - \lambda)^{-d(\lambda)}(0) = H_0((T - \lambda)|_{E_1})$  and  $(T - \lambda)|_{E_2}$  is bounded below, where  $d(\lambda)$  is the order of the left pole at  $\lambda$ .*

*Proof.* The hypotheses imply that  $T - \lambda$  is quasi-Fredholm of order  $d(\lambda)$ , and the closed subspaces  $(T - \lambda)^{-d(\lambda)}(0) + (T - \lambda)\mathcal{X}$  and  $(T - \lambda)^{-1}(0) \cap (T - \lambda)^{d(\lambda)}\mathcal{X}$  are complemented in  $\mathcal{X}$ . Hence, [13, Theorem 5], there exist  $T$ -invariant closed subspaces  $E_1$  and  $E_2$  such that  $\mathcal{X} = E_1 \oplus E_2$ ,  $(T - \lambda)^{d(\lambda)}|_{E_1} = 0$  and  $(T - \lambda)|_{E_2}$  is semi-regular. (Recall, [1, Page 7], that  $T - \lambda$  is semi-regular if  $(T - \lambda)\mathcal{X}$  is closed and  $(T - \lambda)^{-n}(0) \subseteq (T - \lambda)^m\mathcal{X}$  for all natural numbers  $m, n$ .) Since  $\text{asc}(T - \lambda) = d(\lambda) < \infty \iff (T - \lambda)^{d(\lambda)}\mathcal{X} \cap (T - \lambda)^{-n}(0) = \{0\}$  for every natural number  $n$ , the semi-regular operator  $(T - \lambda)|_{E_2}$  is injective. Hence  $(T - \lambda)|_{E_2}$  is bounded below. Observe that

$$\begin{aligned} H_0(T - \lambda) &= H_0((T - \lambda)|_{E_1}) \oplus H_0((T - \lambda)|_{E_2}) \\ &= E_1 \oplus 0 = E_1. \end{aligned}$$

This, since  $H_0(T - \lambda) = (T - \lambda)^{-d(\lambda)}(0)$  by [2, Theorem 2.3], completes the proof.  $\square$

Next follows the main result of this section.

**Theorem 3.2.** *Let  $A$  and  $B$  be left polaroid operators. If  $A$  and  $B$  satisfy property  $(\mathcal{P})$ , or if  $\mathcal{X}$  and  $\mathcal{Y}$  are Hilbert spaces, then  $A \otimes B$  is left polaroid.*

*Proof.* We consider the case in which  $\mathcal{X}, \mathcal{Y}$  are Banach spaces and  $A, B$  satisfy property  $(\mathcal{P})$ ; since  $A, B$  automatically satisfy property  $(\mathcal{P})$  in the case in which  $\mathcal{X}, \mathcal{Y}$  are Hilbert spaces, the proof for the Hilbert space case is a consequence of the Banach space case.

Since  $\sigma_a(A \otimes B) = \sigma_a(A)\sigma_a(B)$ ,  $\text{iso } \sigma_a(A \otimes B) = \text{iso } (\sigma_a(A)\sigma_a(B)) \subseteq \text{iso } \sigma_a(A)\text{iso } \sigma_a(B)$ ; furthermore, this is easily seen,  $\text{iso } \sigma_a(A \otimes B) \setminus \{0\} \subseteq \text{iso } \sigma_a(A)\text{iso } \sigma_a(B) \subseteq \text{iso } \sigma_a(A \otimes B) \cup \{0\}$ . We consider the cases (i)  $0 \neq \lambda \in \text{iso } \sigma_a(A \otimes B)$  and (ii)  $0 = \lambda \in \text{iso } \sigma_a(A \otimes B)$  separately.

(i). In this case, for every  $\lambda \in \text{iso } \sigma_a(A \otimes B)$  there exist non-zero  $\mu \in \text{iso } \sigma_a(A)$  and  $\nu \in \text{iso } \sigma_a(B)$  such that  $\mu\nu = \lambda$ . The operator  $A$  and  $B$  being left polaroid operators which satisfy property  $(\mathcal{P})$ , there exist (by Lemma 3.1)  $A$ -invariant closed subspaces  $M_1$  and  $M_2$ , and  $B$ -invariant closed subspaces  $N_1$  and  $N_2$ , such that

$$\mathcal{X} = M_1 \oplus M_2, \mathcal{Y} = N_1 \oplus N_2, \text{ and } \mathcal{X} \overline{\otimes} \mathcal{Y} = M_1 \overline{\otimes} N_1 \oplus M_1 \overline{\otimes} N_2 \oplus M_2 \overline{\otimes} N_1 \oplus M_2 \overline{\otimes} N_2,$$

where the closed subspaces  $M_i \overline{\otimes} N_j$ ,  $1 \leq i, j \leq 2$ , are  $A \otimes B$ -invariant and, for some integers  $d_1, d_2 \geq 1$ ,

$$(A - \mu I)^{d_1}|_{M_1} = 0 = (B - \nu I)^{d_2}|_{N_1}, \text{ and } (A - \mu I)|_{M_2}, (B - \nu I)|_{N_2} \text{ are bounded below.}$$

Let  $d_1 + d_2 = d$ . Then, since

$$A \otimes B - \lambda(I \otimes I) = (A - \mu I) \otimes B + (\mu I \otimes (B - \nu I)) = S + T \text{ say,}$$

$$\{A \otimes B - \lambda(I \otimes I)\}^d = \sum_{k=0}^d \binom{d}{k} S^k T^{d-k}$$

implies that

$$\{A \otimes B - \lambda(I \otimes I)\}^d|_{M_i \overline{\otimes} N_j} = 0; 1 \leq i, j \leq 2 \text{ and } i, j \neq 2.$$

Furthermore, since  $\mu \notin \sigma_a(A|_{M_2})$  and  $\nu \notin \sigma_a(B|_{N_2})$ ,  $\lambda = \mu\nu \notin \sigma_a(A|_{M_2} \otimes B|_{N_2}) = \sigma_a(A \otimes B|_{M_2 \overline{\otimes} N_2})$ , and hence  $\{A \otimes B - \lambda(I \otimes I)\}^d|_{M_2 \overline{\otimes} N_2}$  is bounded below. Thus  $\mathcal{X} \overline{\otimes} \mathcal{Y}$  is the direct sum of two  $A \otimes B$ -invariant closed subspaces of  $\mathcal{X} \overline{\otimes} \mathcal{Y}$  such that the restriction of  $A \otimes B - \lambda(I \otimes I)$  to one of them is nilpotent and its restriction to the other is bounded below. Apparently,  $\text{asc}(A \otimes B - \lambda(I \otimes I)) \leq d < \infty$  and  $\{A \otimes B - \lambda(I \otimes I)\}^{d+1}(\mathcal{X} \overline{\otimes} \mathcal{Y})$  is closed; hence  $A \otimes B$  is left polar at  $\lambda$ .

(ii). If  $\lambda = 0 \in \text{iso } \sigma_a(A \otimes B)$ , then either (a) 0 is not in one of  $\sigma_a(A)$  and  $\sigma_a(B)$ , or (b)  $0 \in \sigma_a(A) \cap \sigma_a(B)$ . If (a) holds and  $0 \notin \sigma_a(A)$ , then  $0 \in \text{iso } \sigma_a(B)$ ,  $A$  is left invertible and there exist  $B$ -invariant closed subspaces  $N_1$  and  $N_2$  such that  $\mathcal{Y} = N_1 \oplus N_2$ ,  $B|_{N_1}$  is nilpotent and  $B|_{N_2}$  is bounded below. Since  $\mathcal{X} \overline{\otimes} \mathcal{Y} = \mathcal{X} \overline{\otimes} N_1 \oplus \mathcal{X} \overline{\otimes} N_2$ ,  $A \otimes B|_{\mathcal{X} \overline{\otimes} N_1}$  is nilpotent and  $A \otimes B|_{\mathcal{X} \overline{\otimes} N_2}$  is bounded below. Thus  $A \otimes B$  is left polar at 0. Since a similar argument works for the case in which  $0 \in \text{iso } \sigma_a(A)$  and  $0 \notin \sigma_a(B)$ , we are left with case (b). If  $0 \in \sigma_a(A) \cap \sigma_a(B)$ , then either (b<sub>1</sub>)  $0 \in \text{iso } \sigma_a(A) \cap \text{iso } \sigma_b(B)$ , or (b<sub>2</sub>)  $0 \in \text{iso } \sigma_a(A) \cap \text{acc } \sigma_a(B)$ , or (b<sub>3</sub>)  $0 \in \text{acc } \sigma_a(A) \cap \text{iso } \sigma_a(B)$ . If (b<sub>1</sub>) holds, then we copy the argument of (i) above, with  $\mu = \nu = 0$ , to obtain  $A \otimes B$  is left polar at 0. If, instead, (b<sub>2</sub>) (respectively, (b<sub>3</sub>)) holds, then  $\sigma_a(A) = \{0\}$  (respectively,  $\sigma_a(B) = \{0\}$ ), and  $A$  (respectively,  $B$ ) is nilpotent. This implies that  $A \otimes B$  is nilpotent, hence left polaroid.  $\square$

Evidently, Theorem 3.2 has a right polar analogue. Observe that if an operator  $T \in B(\mathcal{X})$  is polaroid (i.e., it is both left and right polaroid), then  $\text{iso } \sigma(T) \cap \{\sigma_{UBW}(T) \cup \sigma_{LBW}(T)\} = \text{iso } \sigma(T) \cap \sigma_{BW}(T) = \emptyset$ . In such a case, there exists an integer  $d(\lambda) > 0$  such that  $\text{co-dim}((T - \lambda)\mathcal{X} + (T - \lambda)^{-d(\lambda)}(0))$  and  $\dim((T - \lambda)^{-1}(0) \cap (T - \lambda)^{d(\lambda)}\mathcal{X})$  are both finite at every  $\lambda \in \text{iso } \sigma(T)$ . Hence there exist  $T$ -invariant closed subspaces  $E_1$  and  $E_2$  such that  $\mathcal{X} = E_1 \oplus E_2$ ,  $(T - \lambda)|_{E_1}$  is  $d(\lambda)$ -nilpotent and  $(T - \lambda)|_{E_2}$  is invertible at every  $\lambda \in \text{iso } \sigma(T)$  (cf. [13, Theorem 7]). The argument of the proof of Theorem 3.2 implies the following.

**Corollary 3.3.** [8, Theorem 3] *A and B polaroid implies  $A \otimes B$  polaroid.*

The Hilbert space version of Theorem 3.2 has a  $\tau_{AB}$  analogue.

**Theorem 3.4.** *If  $A \in B(\mathcal{H})$  and  $B^* \in B(\mathcal{K})$  are left polaroid Hilbert space operators, then  $\tau_{AB}$  is left polaroid*

*Proof.* To prove the Theorem, one argues as in the proof of [8, Corollary 4]:  $B(B(\mathcal{K}), B(\mathcal{H}))$  is an ultraprime Banach  $(B(\mathcal{K}), B(\mathcal{H}))$  bimodule, and hence  $\tau_{AB}$  is just  $A \otimes B^*$ . Here the ultraprime condition  $\|L_A R_B\| = \|A\| \|B\|$  ensures that the operator norm of the bimodule induces a uniform cross-norm on  $\mathcal{H} \overline{\otimes} \mathcal{K}$ .  $\square$

## 4. Finitely left polaroid operators

Recall that upper semi-Fredholm operators have a Kato decomposition: indeed, if  $\lambda \notin \sigma_{ab}(T)$ , then there exist  $T$ -invariant closed subspaces  $E_1$  and  $E_2$  such that  $H_0(T - \lambda) = H_0((T - \lambda)|_{E_1}) = (T - \lambda)^{-d}(0)$  for some integer  $d = \text{asc}(T - \lambda) > 0$ ,  $\dim H_0(T - \lambda) = \dim E_1 < \infty$ , and  $(T - \lambda)|_{E_2}$  is bounded below. Apparently, if the operators  $A$  and  $B$  are finitely left polaroid, then  $A \otimes B$  is left polaroid. The finitely left polaroid property does not transfer from  $A, B$  to  $A \otimes B$ ; the problem, as one would expect, lies with the point 0.

**Theorem 4.1.** *If  $A$  and  $B$  are finitely left polaroid, then  $A \otimes B$  is finitely left polaroid if and only if  $0 \notin \text{iso } \sigma_a(A \otimes B)$ .*

*Proof.* Recall, [7, Lemma 5], that

$$\sigma_{ab}(A \otimes B) = \sigma_{ab}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{ab}(B).$$

Suppose that  $0 \neq \lambda \notin \text{iso } \sigma_a(A \otimes B)$ . Then there exist non-zero  $\mu \in \text{iso } \sigma_a(A)$  and  $\nu \in \text{iso } \sigma_a(B)$  such that  $\mu\nu = \lambda$ . If  $A$  and  $B$  are finitely left polaroid, then  $\mu \notin \sigma_{ab}(A)$  and  $\nu \notin \sigma_{ab}(B)$ . Hence

$$\lambda \notin \sigma_{ab}(A \otimes B) \iff \lambda \in \Pi_0^l(A \otimes B).$$

Now let  $\lambda = 0$ . Since the finitely left polaroid hypothesis on  $A$  (respectively,  $B$ ) implies that  $\mathcal{X}$  (respectively,  $\mathcal{Y}$ ) has a direct sum decomposition of the type considered in the proof of Theorem 3.2 whenever  $0 \in \text{iso } \sigma_a(A)$  (respectively,  $0 \in \text{iso } \sigma_a(B)$ ), it follows from the argument of the proof of Theorem 3.2, see (ii) of the proof, that  $A \otimes B$  is left polaroid at 0, with  $\alpha(A \otimes B) = \infty$ . Hence  $A \otimes B$  is not finitely left polaroid at 0.  $\square$

**Remark 4.2.** The upper semi-Fredholm spectrum  $\sigma_{SF_+}(T)$  of an operator  $T$  satisfies the inclusion  $\sigma_{SF_+}(T) \subseteq \sigma_{ab}(T)$ . Since  $0 \notin \sigma_a(T) \setminus \sigma_{SF_+}(T)$  for every operator  $T$  ([7, Lemma 4]),  $0 \notin \sigma_a(A \otimes B) \setminus \sigma_{ab}(A \otimes B) = \Pi_0^\ell(A \otimes B)$ : this provides an alternative proof of a part of Theorem 4.1.

**Remark 4.3.** If the hypotheses of Theorem 4.1 are satisfied, then to prove Theorem 4.1 it is enough to consider the case  $0 \in \sigma_a(A \otimes B)$ ; see Lemma 2.2(i) and Remark 2.1(iii). Observe that if  $0 \notin \text{iso } \sigma_a(A \otimes B)$ , then  $A \otimes B$  is finitely left polaroid (by Lemma 2.2(i) and Remark 2.1(iii)). If, instead,  $A \otimes B$  is finitely left polaroid and  $0 \in \text{iso } \sigma_a(A \otimes B) \subseteq \text{iso } \sigma_a(A) \text{iso } \sigma_a(B) = \Pi_0^l(A) \Pi_0^l(B)$ , then  $0 \in \Pi_0^l(A)$  or  $0 \in \Pi_0^l(B)$ . However, if  $0 \in \Pi_0^l(A)$  (respectively,  $0 \in \Pi_0^l(B)$ ), then, since  $\text{acc } \sigma_a(B) = \sigma_{ab}(B) \neq \emptyset$  (respectively,  $\text{acc } \sigma_a(A) = \sigma_{ab}(A) \neq \emptyset$ ) and  $\sigma_a(A \otimes B) = \sigma_a(A)\sigma_a(B)$ ,  $0 \in \text{acc } \sigma_a(A \otimes B)$ , which is a contradiction. This provides yet another proof of Theorem 4.1.

The following remark is a supplement to the conclusions of Theorem 4.1. In fact, given  $A$  and  $B$  two finitely left polaroid operators,  $\sigma_a(A \otimes B)$  will be fully described in terms of the Browder essential approximate point spectrum and the set of finite left poles of the operators  $A$  and  $B$ . Observe that Remark 2.3 describes  $\sigma_{ab}(A \otimes B)$  for finitely left polaroid operators  $A$  and  $B$ .

**Remark 4.4.** Let, as in Theorem 4.1,  $A$  and  $B$  be two finitely left polaroid operators.

(i) If  $0 \notin \sigma_a(A) \cdot \sigma_a(B) = \sigma_a(A \otimes B)$ , then according to Theorem 4.1 and Lemma 2.2(ii),  $\text{acc } \sigma_a(A \otimes B) = \sigma_{ab}(A \otimes B)$  and  $\Pi_0^l(A \otimes B) = \text{iso } \sigma_a(A \otimes B) = \Pi_0^l(A) \cdot \Pi_0^l(B)$ . Same conclusions can be derived when  $0 \in \text{acc } \sigma_a(A) \setminus \text{iso } \sigma_a(B)$  or  $0 \in \text{acc } \sigma_a(B) \setminus \text{iso } \sigma_a(A)$ .

(ii) If  $0 \in \text{acc } \sigma_a(A) \cap \text{iso } \sigma_a(B)$ , then according to the last observation in Remark 2.1 and Lemma 2.2(i),  $\text{acc } \sigma(A \otimes B) = \sigma_{ab}(A \otimes B)$ . In addition, according to Lemma 2.2(ii),  $\text{iso } \sigma_a(A \otimes B) = \Pi_0^l(A \otimes B) = \Pi_0^l(A) \cdot (\Pi_0^l(B) \setminus \{0\})$ . Similarly, if  $0 \in \text{acc } \sigma_a(B) \cap \text{iso } \sigma_a(A)$ , then  $\text{iso } \sigma_a(A \otimes B) = \Pi_0^l(A \otimes B) = (\Pi_0^l(A) \setminus \{0\}) \cdot \Pi_0^l(B)$ .

(iii) If  $0 \in \text{iso } \sigma_a(A)$  and  $0 \notin \sigma_a(B)$ , then since  $\sigma_a(A) \cdot \sigma_a(B) = \sigma_a(A \otimes B)$ , a standard argument on convergent subsequences proves that  $0 \in \text{iso } \sigma_a(A \otimes B)$ . Consequently, according to Lemma 2.2(i)-(ii) and [7, Lemma 5],  $\sigma_{ab}(A \otimes B) = \text{acc } \sigma_a(A \otimes B) \cup \{0\}$ ,  $I_0^a(A \otimes B) = \{0\}$ ,  $\Pi_0^l(A \otimes B) = (\Pi_0^l(A) \setminus \{0\}) \cdot \Pi_0^l(B)$ ,  $\text{iso } \sigma_a(A \otimes B) = \Pi_0^l(A) \cdot \Pi_0^l(B)$  and  $\text{acc } \sigma_a(A \otimes B) = \sigma_{ab}(A) \cdot \sigma_{ab}(B) \cup \sigma_{ab}(A) \cdot \Pi_0^l(B) \cup (\Pi_0^l(A) \setminus \{0\}) \cdot \sigma_{ab}(B)$ .

(iv) If  $0 \in \text{iso } \sigma_a(A) \cap \text{iso } \sigma_a(B)$ , then an argument similar to the one in (iii) proves that  $\text{iso } \sigma_a(A \otimes B) = \Pi_0^l(A) \cdot \Pi_0^l(B)$ ,  $\Pi_0^l(A \otimes B) = (\Pi_0^l(A) \setminus \{0\}) \cdot (\Pi_0^l(B) \setminus \{0\})$ ,  $I_0^a(A \otimes B) = \{0\}$ ,  $\sigma_{ab}(A \otimes B) = \text{acc } \sigma_a(A \otimes B) \cup \{0\}$  and  $\text{acc } \sigma_a(A \otimes B) = \sigma_{ab}(A) \cdot \sigma_{ab}(B) \cup \sigma_{ab}(A) \cdot (\Pi_0^l(B) \setminus \{0\}) \cup (\Pi_0^l(A) \setminus \{0\}) \cdot \sigma_{ab}(B)$ .

Note that the transfer property for finitely left polaroid operators holds in (i) and (ii).

We consider next the elementary operator  $\tau_{AB}$  (where, as before,  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$ ).

**Theorem 4.5.** *If  $A$  and  $B^*$  are finitely left polaroid operators, then  $\tau_{AB}$  is finitely left polaroid if and only if  $0 \notin \text{iso } \sigma_a(\tau_{AB})$ .*

*Proof.* Recall that  $\sigma_a(\tau_{AB}) = \sigma_a(A)\sigma_a(B^*)$  and  $\sigma_{ab}(\tau_{AB}) = \sigma_{ab}(A)\sigma_a(B^*) \cup \sigma_a(A)\sigma_{ab}(B^*)$  [6, Proposition 4.1]. Now argue as in the proof of Theorem 4.1 to prove that  $\tau_{AB}$  is finitely left polaroid at every non-zero  $\lambda \in \text{iso } \sigma_a(\tau_{AB})$ , and as in Remark 4.2 to prove that  $\tau_{AB}$  is not finitely left polaroid at  $0 \in \text{iso } \sigma_a(\tau_{AB})$ .  $\square$

Apparently, an alternative proof of Theorem 4.5 is obtained from an argument similar to the one in Remark 4.3. Furthermore, arguing just as for the operator  $A \otimes B$  in Remark 4.4, it is possible to obtain a complete characterization of the sets  $\sigma_a(\tau_{AB})$ ,  $\text{acc } \sigma_a(\tau_{AB})$ ,  $\text{iso } \sigma_a(\tau_{AB})$ ,  $I_0^a(\tau_{AB})$  and  $\Pi_0^l(\tau_{AB})$ , in terms of the corresponding sets for  $A$  and  $B^*$ . The details are left to the reader.

We end this section by studying perturbations of finitely left polaroid operators by quasi-nilpotents.

If  $Q_1 \in B(\mathcal{X})$  and  $Q_2 \in B(\mathcal{Y})$  are quasi-nilpotents which commute with  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  respectively, then  $(A+Q_1) \otimes (B+Q_2) = (A \otimes B) + Q$ , where  $Q = A \otimes Q_2 + Q_1 \otimes B + Q_1 \otimes Q_2$  is a quasi-nilpotent which commutes with  $A \otimes B$ . Since

$$\sigma_a((A+Q_1) \otimes (B+Q_2)) = \sigma_a(A+Q_1)\sigma_a(B+Q_2) = \sigma_a(A)\sigma_a(B) \text{ and}$$

$$\begin{aligned} \sigma_{ab}((A+Q_1) \otimes (B+Q_2)) &= \sigma_{ab}(A+Q_1)\sigma_a(B+Q_2) \cup \sigma_a(A+Q_1)\sigma_{ab}(B+Q_2) \\ &= \sigma_{ab}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{ab}(B), \end{aligned}$$

$A$  and  $B$  finitely left polaroid implies  $(A+Q_1) \otimes (B+Q_2)$  finitely left polaroid at every  $0 \neq \lambda \in \text{iso } \sigma_a((A+Q_1) \otimes (B+Q_2))$ . Furthermore, since  $A \otimes B = (A+Q_1) \otimes (B+Q_2) - Q$ ,  $Q$  as above, we have:

**Corollary 4.6.** *If  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  are finitely left polaroid, and  $Q_1 \in B(\mathcal{X})$  and  $Q_2 \in B(\mathcal{Y})$  are quasi-nilpotents which commute with  $A$  and  $B$  respectively, then  $(A+Q_1) \otimes (B+Q_2)$  is finitely left polaroid if and only if  $0 \notin \text{iso } \sigma_a((A+Q_1) \otimes (B+Q_2))$ .*

## 5. An application

For an operator  $T \in B(\mathcal{X})$ , let  $E^a(T) = \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(T - \lambda)\}$  and  $E_0^a(T) = \{\lambda \in E^a(T) : \alpha(T - \lambda) < \infty\}$ . Recall that  $T$  is said to satisfy  $a$ -Browder's theorem,  $a$ -Bt for short (respectively, generalized  $a$ -Browder's theorem,  $a$ -gBt for short) if  $\sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_0^l(T)$  (respectively,  $\sigma_a(T) \setminus \sigma_{UBW}(T) = \Pi^\ell(T)$ ). The following equivalence is well known [4, Theorem 2.2]:  $T$  satisfies  $a$ -Bt if and only if  $T$  satisfies  $a$ -gBt.  $T$  satisfies  $a$ -Weyl's theorem,  $a$ -Wt for short (respectively, generalized  $a$ -Weyl's theorem,  $a$ -gWt for short) if  $\sigma_a(T) \setminus \sigma_{aw}(T) = E_0^a(T)$  (respectively,  $\sigma_a(T) \setminus \sigma_{UBW}(T) = E^a(T)$ ). The following one way implication holds:  $T$  satisfies  $a$ -gWt implies  $T$  satisfies  $a$ -Wt. Next generalized  $a$ -Weyl's theorem for  $A \otimes B$  will be studied under the assumption  $A$  and  $B$  (finitely) left polaroid.



**Theorem 5.1.** *Suppose that  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  satisfy  $a$ -Bt. If (i)  $A, B$  are finitely left polaroid, or (ii)  $\mathcal{X}, \mathcal{Y}$  are Hilbert spaces and  $A, B$  are left polaroid, then  $A \otimes B$  satisfies  $a$ -gWt if and only if  $\sigma_{aw}(A \otimes B) = \sigma_{aw}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{aw}(B)$ .*

*Proof.* If  $A$  and  $B$  satisfy  $a$ -Bt, then  $A \otimes B$  satisfies  $a$ -Bt (consequently, also  $a$ -gBt) if and only if  $\sigma_{aw}(A \otimes B) = \sigma_{aw}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{aw}(B)$  ([7, Theorem 1]). Thus  $\sigma_a(A \otimes B) \setminus \sigma_{UBW}(A \otimes B) = \Pi^\ell(A \otimes B) \subseteq E^a(A \otimes B)$ . Since either of the hypotheses (i) and (ii) of the statement of the theorem implies  $A \otimes B$  is left polaroid,  $E^a(A \otimes B) \subseteq \Pi^\ell(A \otimes B)$ . Hence  $A \otimes B$  satisfies  $a$ -gWt. The necessity being obvious from the implications  $A \otimes B$  satisfies  $a$ -gWt implies  $A \otimes B$  satisfies  $a$ -gBt implies  $A \otimes B$  satisfies  $a$ -Bt, the proof is complete.  $\square$

The finite left polaroid requirement in Theorem 5.1 may be relaxed in the case in which  $A^*$  has SVEP on  $\Pi^\ell(A)$  and  $B^*$  has SVEP on  $\Pi^\ell(B)$ .

**Theorem 5.2.** *Suppose that  $A \in B(\mathcal{X})$  and  $B \in B(\mathcal{Y})$  satisfy  $a$ -Bt. If  $A^*$  has SVEP at points not in  $\sigma_{UBW}(A)$ ,  $B^*$  has SVEP at points not in  $\sigma_{UBW}(B)$  and  $A, B$  are left polaroid, then  $A \otimes B$  satisfies  $a$ -gWt if and only if  $\sigma_{aw}(A \otimes B) = \sigma_{aw}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{aw}(B)$ .*

*Proof.* The necessity follows as in the proof of Theorem 5.1; also, if  $A, B$  satisfy  $a$ -Bt, and  $\sigma_{aw}(A \otimes B) = \sigma_{aw}(A)\sigma_a(B) \cup \sigma_a(A)\sigma_{aw}(B)$ , then  $A \otimes B$  satisfies  $a$ -gBt. If  $\mu \in \Pi^\ell(A)$ , then the SVEP hypothesis on  $A^*$  implies the existence of  $A$ -invariant subspaces  $M_1$  and  $M_2$  such that  $\mathcal{X} = M_1 \oplus M_2$ ,  $(A - \mu)|_{M_1}$  is nilpotent and  $(A - \mu)|_{M_2}$  is invertible (see the argument preceding Corollary 3.3); similarly, if  $\nu \in \Pi^\ell(B)$ , then the SVEP hypothesis on  $B^*$  implies the existence of  $B$ -invariant subspaces  $N_1$  and  $N_2$  such that  $\mathcal{Y} = N_1 \oplus N_2$ ,  $(B - \nu)|_{N_1}$  is nilpotent and  $(B - \nu)|_{N_2}$  is invertible. The slight changes in the argument in the case in which one of  $\mu$  and  $\nu$  is 0 and the other is not a left pole being obvious, it follows from the argument of the proof of Theorem 3.2 that the left polaroid property transfers from  $A$  and  $B$  to  $A \otimes B$ . Hence, see the proof of Theorem 5.1,  $A \otimes B$  satisfies  $a$ -gWt.  $\square$

Evidently, the operator  $A \otimes B$  of Theorem 5.2 satisfies (generalized Weyl's theorem,  $\sigma(A \otimes B) \setminus \sigma_{BW}(A \otimes B) = E(A \otimes B) = \{\lambda \in \text{iso } \sigma(A \otimes B) : \lambda \text{ is an eigenvalue of } A \otimes B\}$  and)  $a$ -Wt. More is true:  $\sigma_a(A \otimes B) \setminus \sigma_{UBW}(A \otimes B) = E(A \otimes B)$ . To see this, we observe that if the hypotheses of Theorem 5.2 are satisfied, then  $\Pi^\ell(A) = \Pi(A) = E(A)$ ,  $\Pi^\ell(B) = \Pi(B) = E(B)$  and  $\sigma_a(A \otimes B) \setminus \sigma_{UBW}(A \otimes B) = \Pi^\ell(A \otimes B) = E^a(A \otimes B)$ . Evidently,  $E(A \otimes B) \subseteq E^a(A \otimes B)$ . Let  $\lambda \in E^a(A \otimes B)$ . If  $\lambda \neq 0$ , then there exists  $\mu \in \text{iso } \sigma_a(A)$  and  $\nu \in \sigma_a(B)$  such that  $\mu \in \Pi^\ell(A) = \Pi(A) \subseteq E(A)$  and  $\nu \in \Pi^\ell(B) = \Pi(B) \subseteq E(B)$ ; hence  $\lambda \in E(A \otimes B)$ . If, instead,  $\lambda = 0$ , then either  $0 \in \Pi^\ell(A) \cap \Pi^\ell(B) = \Pi(A) \cap \Pi(B)$ , or 0 is in one of  $\Pi(A)$ ,  $\Pi(B)$  and not in the other; in either case  $0 \in E(A \otimes B)$ . Hence  $E(A \otimes B) \subseteq E^a(A \otimes B)$ .

Theorem 5.1 has a  $\tau_{AB}$  analogue.

**Corollary 5.3.** *Suppose that  $A \in B(\mathcal{X})$  and  $B^* \in B(\mathcal{Y}^*)$  satisfy  $a$ -Bt. If (i)  $A, B^*$  are finitely left polaroid, or (ii)  $\mathcal{X}, \mathcal{Y}$  are Hilbert spaces and  $A, B^*$  are left polaroid, then  $\tau_{AB}$  satisfies  $a$ -gWt if and only if  $\sigma_{aw}(\tau_{AB}) = \sigma_{aw}(A)\sigma_a(B^*) \cup \sigma_a(A)\sigma_{aw}(B^*)$ .*

*Proof.* To prove the corollary one argues as in the theorem above, using Theorem 3.4 and the fact that if  $A$  and  $B^*$  satisfy  $a$ -Bt then  $\tau_{AB}$  satisfies  $a$ -gBt if and only if  $\sigma_{aw}(\tau_{AB}) = \sigma_{aw}(A)\sigma_a(B^*) \cup \sigma_a(A)\sigma_{aw}(B^*)$  ([6, Theorem 4.5]).  $\square$

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